Normal Vectors on Modified Hopf Manifolds of Delay Differential Equations

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Abstract

This document states the normal vector system for modified Hopf boundaries of delay differential systems with state and parameter dependent delays. Specifically, this document states the proof for Proposition 1 in the paper entitled *Robust optimization of delay differential equations with state and parameter dependent delays* by J. Otten and M. Mönnigmann, which has been accepted for the 55$^{\text{th}}$ IEEE Conference on Decision and Control [1].

1 Introduction

1.1 System Class

We consider continuous time systems with uncertain and state dependent delays

$$\dot{x}(t) = f(x(t), x(t-\tau_1), \ldots, x(t-\tau_m), \alpha) \tag{1}$$

with state vector $x \in \mathbb{R}^n$, uncertain parameters $\alpha \in \mathbb{R}^{n_\alpha}$, m delays τ_i and a smooth f that maps from $\mathbb{R}^{n(m+1)} \times \mathbb{R}^{n_\alpha}$, or an open subset thereof, into $\mathbb{R}^n$. The delays may be functions of the current state and uncertain parameters,

$$\tau_i = \tau_i(x(t), \alpha)\,. \tag{2}$$

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1.2 Preliminaries

The solution to a set of nonlinear equations will be called regular, if the Jacobian of the equations evaluated at this solution has full rank.

We introduce the following abbreviations.

$$\mathrm{s}(\sigma,\omega,\tau) = \exp(-\sigma\tau)\sin(\omega\tau)$$
$$\mathrm{c}(\sigma,\omega,\tau) = \exp(-\sigma\tau)\cos(\omega\tau)\,.$$

Furthermore, we introduce the delay $\tau_0 = 0$ to simplify the notation. This permits to replace $x(t)$ by $x(t-\tau_0)$ and state expressions for $x(t-\tau_i)$, $i = 0,\dots,m$ instead of for $x(t)$ and for $x(t-t_i)$, $i = 1,\dots,m$ separately.

The Jacobian of the right hand side of (1) w.r.t $x(t-\tau_i)$ are denoted by A_i. All Jacobians are assumed to be evaluated at a steady state if not noted otherwise. We denote such a steady state $\tilde{x}$.

We refer to $\lambda \in \mathbb{C}$ as an eigenvalue at a steady state $\tilde{x}$ of (1), if

$$\det\left(\lambda I - A_0 - \sum_{i=1}^{m} A_i \exp(-\lambda\tau_i)\right) = 0 \tag{4}$$

(cf. [2]).

By *modified Hopf point* we refer to a steady state that has a leading complex conjugate pair of eigenvalues with nonzero real part. A manifold of such points is called modified Hopf manifold.

2 Augmented system of Modified Hopf Manifold

Lemma 1 (augmented system modified Hopf [1])**.** *Assume $\tilde{x}^{(c)}$ is a steady state for parameter values $\alpha^{(c)}$. If $\lambda = \sigma \pm \mathrm{i}\omega$ are the leading eigenvalues at the steady state $\tilde{x}^{(c)}$, then there exist vectors $a, b \in \mathbb{R}^n$ such that $(\tilde{x}^{(c)}, \alpha^{(c)})$, obey the equations*

$$f(\tilde{x}^{(c)}, \tilde{x}^{(c)}, \dots, \tilde{x}^{(c)}, \alpha^{(c)}) = 0 \tag{5a}$$

$$\sigma a - \omega b - \sum_{i=0}^{m} A_i\big(\mathrm{c}(\sigma,\omega,\tau_i)a + \mathrm{s}(\sigma,\omega,\tau_i)b\big) = 0 \tag{5b}$$

$$\omega a + \sigma b - \sum_{i=0}^{m} A_i\big(\mathrm{c}(\sigma,\omega,\tau_i)b - \mathrm{s}(\sigma,\omega,\tau_i)a\big) = 0 \tag{5c}$$

$$a'a + b'b - 1 = 0 \tag{5d}$$

$$a'b = 0\,. \tag{5e}$$

3 Normal Vector on Modified Hopf Manifold

The following proposition is stated in [1], but only a sketch of the proof was given there. It is the purpose of the present document to state a complete proof.

Proposition 1 (normal vectors modified Hopf)**.** *If* $(\tilde{x}^{(c)}, \alpha^{(c)}, \omega, a, b)$ *is a regular solution to* (5) *for an arbitrary but fixed* σ*, then* r *that obeys the following equations is normal to the manifold of modified Hopf points at this solution:*

$$\textit{equations } (5) \tag{6a}$$

$$\begin{bmatrix} \nabla_{\tilde{x}^{(c)}} f' & B_{12} & B_{13} & 0 & 0 \\ 0 & B_{22} & B_{23} & 2a & b \\ 0 & B_{32} & B_{33} & 2b & a \\ 0 & B_{42} & B_{43} & 0 & 0 \end{bmatrix} \kappa = 0 \tag{6b}$$

$$\begin{bmatrix} \nabla_{\alpha^{(c)}} f' & B_{52} & B_{53} & 0 & 0 \end{bmatrix} \kappa - r = 0 \tag{6c}$$

$$r'r - 1 = 0 \tag{6d}$$

where

$$\begin{aligned} B_{12} =& \sum_{i=0}^{m} \sigma(\nabla_{\tilde{x}^{(c)}} \tau_i)\big(\mathrm{c}(\sigma,\omega,\tau_i)a' + \mathrm{s}(\sigma,\omega,\tau_i)b'\big)A_i' \\ &- \sum_{i=0}^{m} \omega(\nabla_{\tilde{x}^{(c)}} \tau_i)\big(\mathrm{c}(\sigma,\omega,\tau_i)b' - \mathrm{s}(\sigma,\omega,\tau_i)a'\big)A_i' \\ &- \sum_{i=0}^{m} \mathrm{c}(\sigma,\omega,\tau_i)(\nabla_{\tilde{x}^{(c)}} a'A_i') + \mathrm{s}(\sigma,\omega,\tau_i)(\nabla_{\tilde{x}^{(c)}} b'A_i')\,, \\ B_{13} =& \sum_{i=0}^{m} \sigma(\nabla_{\tilde{x}^{(c)}} \tau_i)\big(\mathrm{c}(\sigma,\omega,\tau_i)b' - \mathrm{s}(\sigma,\omega,\tau_i)a'\big)A_i' \\ &+ \sum_{i=0}^{m} \omega(\nabla_{\tilde{x}^{(c)}} \tau_i)\big(\mathrm{s}(\sigma,\omega,\tau_i)b' + \mathrm{c}(\sigma,\omega,\tau_i)a'\big)A_i' \\ &- \sum_{i=0}^{m} \mathrm{c}(\sigma,\omega,\tau_i)(\nabla_{\tilde{x}^{(c)}} b'A_i') - \mathrm{s}(\sigma,\omega,\tau_i)(\nabla_{\tilde{x}^{(c)}} a'A_i')\,, \end{aligned}$$

$$B_{22} = \sigma I - \sum_{i=0}^{m} \mathrm{c}(\sigma,\omega,\tau_i)A_i'\,,$$

$$B_{23} = \omega I + \sum_{i=0}^{m} \mathrm{s}(\sigma,\omega,\tau_i) A_i' ,$$

$$\begin{aligned} B_{32} =& -\omega I - \sum_{i=0}^{m} \mathrm{s}(\sigma,\omega,\tau_i) A_i' , \\ B_{33} =& \sigma I - \sum_{i=0}^{m} \mathrm{c}(\sigma,\omega,\tau_i) A_i' , \end{aligned}$$

$$\begin{aligned} B_{42} =& -b' + \sum_{i=0}^{m} \tau_i \big(\mathrm{s}(\sigma,\omega,\tau_i) a' - \mathrm{c}(\sigma,\omega,\tau_i) b'\big) A_i' , \\ B_{43} =& a' + \sum_{i=0}^{m} \tau_i \big(\mathrm{s}(\sigma,\omega,\tau_i) b' + \mathrm{c}(\sigma,\omega,\tau_i) a'\big) A_i' , \end{aligned}$$

$$\begin{aligned} B_{52} =& \sum_{i=0}^{m} \sigma (\nabla_{\alpha^{(c)}} \tau_i) \big(\mathrm{c}(\sigma,\omega,\tau_i) a' + \mathrm{s}(\sigma,\omega,\tau_i) b'\big) A_i' \\ & + \sum_{i=0}^{m} \omega (\nabla_{\alpha^{(c)}} \tau_i) \big(\mathrm{s}(\sigma,\omega,\tau_i) a' - \mathrm{c}(\sigma,\omega,\tau_i) b'\big) A_i' \\ & - \sum_{i=0}^{m} \mathrm{c}(\sigma,\omega,\tau_i) (\nabla_{\alpha^{(c)}} a' A_i') + \mathrm{s}(\sigma,\omega,\tau_i) (\nabla_{\alpha^{(c)}} b' A_i') , \\ B_{53} =& \sum_{i=0}^{m} \sigma (\nabla_{\alpha^{(c)}} \tau_i) \big(\mathrm{c}(\sigma,\omega,\tau_i) b' - \mathrm{s}(\sigma,\omega,\tau_i) a'\big) A_i' \\ & + \sum_{i=0}^{m} \omega (\nabla_{\alpha^{(c)}} \tau_i) \big(\mathrm{s}(\sigma,\omega,\tau_i) b' + \mathrm{c}(\sigma,\omega,\tau_i) a'\big) A_i' \\ & - \sum_{i=0}^{m} \mathrm{c}(\sigma,\omega,\tau_i) (\nabla_{\alpha^{(c)}} b' A_i') - \mathrm{s}(\sigma,\omega,\tau_i) (\nabla_{\alpha^{(c)}} a' A_i') . \end{aligned}$$

The expressions $\nabla_{\tilde{x}^{(c)}} a' A_i'$ *are given by*

$$(\nabla_{\tilde{x}^{(c)}} a' A_i')_{\mu,\nu} = \sum_{\rho=1}^{n} a_\rho \frac{\partial^2 f_\nu}{\partial \tilde{x}_\mu^{(c)} \, \partial \tilde{x}_\rho^{(c)} (t-\tau_i)} .$$

The expressions $\nabla_{\tilde{x}^{(c)}} b' A_i'$, $\nabla_{\alpha^{(c)}} a' A_i'$ *and* $\nabla_{\alpha^{(c)}} b' A_i'$ *are defined accordingly.*

Proof. Consider (5) as $3n+2$ equations in the $3n+n_\alpha+1$ variables $\tilde{x}^{(c)}$, $\alpha^{(c)}$, ω, a and b. In the neighborhood of a regular solution $(\tilde{x}^{(c)}, \alpha^{(c)}, \omega, a, b) \in \mathbb{R}^n \times \mathbb{R}^{n_\alpha} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$, these equations locally define an $(n_\alpha - 1)$-dimensional manifold of modified Hopf points.

The rows of the Jacobian of (5) w.r.t. $\tilde{x}^{(c)}$, b, a, ω and $\alpha^{(c)}$ at a regular solution span the normal space to the manifold (cf. [3]). It is more convenient to work with the transposed Jacobian in the sequel than with the Jacobian itself. The transposed Jacobian is denoted by B and reads

$$
\begin{aligned}
B = & \begin{bmatrix} B_{11} & B_{12} & B_{13} & B_{14} & B_{15} \\ B_{21} & B_{22} & B_{23} & B_{24} & B_{25} \\ B_{31} & B_{32} & B_{33} & B_{34} & B_{35} \\ B_{41} & B_{42} & B_{43} & B_{44} & B_{45} \\ B_{51} & B_{52} & B_{53} & B_{54} & B_{55} \end{bmatrix} \\
= & \begin{bmatrix} \nabla_{\tilde{x}^{(c)}} \\ \nabla_a \\ \nabla_b \\ \nabla_\omega \\ \nabla_{\alpha^{(c)}} \end{bmatrix} \begin{bmatrix} f(\tilde{x}^{(c)}, \tilde{x}^{(c)}, ..., \tilde{x}^{(c)}, \alpha^{(c)}) \\ \sigma a - \omega b - \sum_{i=0}^m A_i \big(\mathrm{c}(\sigma, \omega, \tau_i) a + \mathrm{s}(\sigma, \omega, \tau_i) b\big) \\ \omega a + \sigma b - \sum_{i=0}^m A_i \big(\mathrm{c}(\sigma, \omega, \tau_i) b - \mathrm{s}(\sigma, \omega, \tau_i) a\big) \\ a'a + b'b - 1 \\ a'b \end{bmatrix}' .
\end{aligned}
$$

The first, fourth and fifth column of B contain the Jacobians of $f'(\tilde{x}^{(c)}, \tilde{x}^{(c)}, ..., \tilde{x}^{(c)}, \alpha^{(c)})$, $a'a + b'b - 1$ and $a'b$, respectively. They result from simple calculations which are not detailed here.

The calculations required to determine B_{22}, B_{23}, B_{32} and B_{33}, which correspond to the Jacobians of $\sigma a' - \omega b' - \sum_{i=0}^m \big(\mathrm{c}(\sigma, \omega, \tau_i) a' - \mathrm{s}(\sigma, \omega, \tau_i) b'\big) A_i'$ and $\omega a' + \sigma b' - \sum_{i=0}^m \big(\mathrm{c}(\sigma, \omega, \tau_i) b' + \mathrm{s}(\sigma, \omega, \tau_i) a'\big) A_i'$ with respect to a, b, and ω, are also simple. They result in the following expressions:

$$
\begin{aligned}
B_{22} = & \nabla_a \left(\sigma a' - \omega b' - \sum_{i=0}^m \big(\mathrm{c}(\sigma, \omega, \tau_i) a' - \mathrm{s}(\sigma, \omega, \tau_i) b'\big) A_i' \right) \\
= & \sigma I - \sum_{i=0}^m \mathrm{c}(\sigma, \omega, \tau_i) A_i' \\
B_{23} = & \nabla_a \left(\omega a' + \sigma b' - \sum_{i=0}^m \big(\mathrm{c}(\sigma, \omega, \tau_i) b' + \mathrm{s}(\sigma, \omega, \tau_i) a'\big) A_i' \right) \\
= & \omega I + \sum_{i=0}^m \mathrm{s}(\sigma, \omega, \tau_i) A_i' ,
\end{aligned}
$$

$$
B_{32} = \nabla_b \left(\sigma a' - \omega b' - \sum_{i=0}^m \big(\mathrm{c}(\sigma, \omega, \tau_i) a' - \mathrm{s}(\sigma, \omega, \tau_i) b'\big) A_i' \right)
$$

$$
\begin{aligned}
&= - \omega I - \sum_{i=0}^{m} \mathrm{s}(\sigma, \omega, \tau_i) A_i' \\
B_{33} &= \nabla_b \left(\omega a' + \sigma b' - \sum_{i=0}^{m} \big(\mathrm{c}(\sigma, \omega, \tau_i) b' + \mathrm{s}(\sigma, \omega, \tau_i) a' \big) A_i' \right) \\
&= \sigma I - \sum_{i=0}^{m} \mathrm{c}(\sigma, \omega, \tau_i) A_i' \,,
\end{aligned}
$$

$$
\begin{aligned}
B_{42} &= \nabla_\omega \left(\sigma a' - \omega b' - \sum_{i=0}^{m} \big(\mathrm{c}(\sigma, \omega, \tau_i) a' - \mathrm{s}(\sigma, \omega, \tau_i) b' \big) A_i' \right) \\
&= - b' + \sum_{i=0}^{m} \tau_i \big(\mathrm{s}(\sigma, \omega, \tau_i) a' - \mathrm{c}(\sigma, \omega, \tau_i) b' \big) \big) A_i' \\
B_{43} &= \nabla_\omega \left(\omega a' + \sigma b' - \sum_{i=0}^{m} \big(\mathrm{c}(\sigma, \omega, \tau_i) b' + \mathrm{s}(\sigma, \omega, \tau_i) a' \big) A_i' \right) \\
&= a' + \sum_{i=0}^{m} \tau_i \big(\mathrm{s}(\sigma, \omega, \tau_i) b' + \mathrm{c}(\sigma, \omega, \tau_i) a' \big) \big) A_i' \,,
\end{aligned}
$$

The Jacobians w.r.t $\tilde{x}^{(c)}$ and $\alpha^{(c)}$ call for more attention. Recall that τ_i is, as described in (2), a function of $\tilde{x}^{(c)}$ and $\alpha^{(c)}$. In this case, the product rule has to be applied to three expression depending on $\tilde{x}^{(c)}$, the exponential function, a trigonometric function and A_i'.

$$
\begin{aligned}
B_{12} &= \nabla_{\tilde{x}^{(c)}} \left(\sigma a' - \omega b' - \sum_{i=0}^{m} \big(\mathrm{c}(\sigma, \omega, \tau_i) a' - \mathrm{s}(\sigma, \omega, \tau_i) b' \big) A_i' \right) \\
&= \sum_{i=0}^{m} \sigma (\nabla_{\tilde{x}^{(c)}} \tau_i) \big(\mathrm{c}(\sigma, \omega, \tau_i) a' + \mathrm{s}(\sigma, \omega, \tau_i) b' \big) A_i' \\
&\quad - \sum_{i=0}^{m} \omega (\nabla_{\tilde{x}^{(c)}} \tau_i) \big(\mathrm{c}(\sigma, \omega, \tau_i) b' - \mathrm{s}(\sigma, \omega, \tau_i) a' \big) A_i' \\
&\quad - \sum_{i=0}^{m} \mathrm{c}(\sigma, \omega, \tau_i) (\nabla_{\tilde{x}^{(c)}} a' A_i') + \mathrm{s}(\sigma, \omega, \tau_i) (\nabla_{\tilde{x}^{(c)}} b' A_i')) \\
B_{13} &= \nabla_{\tilde{x}^{(c)}} \left(\omega a' + \sigma b' - \sum_{i=0}^{m} \big(\mathrm{c}(\sigma, \omega, \tau_i) b' + \mathrm{s}(\sigma, \omega, \tau_i) a' \big) A_i' \right) \\
&= \sum_{i=0}^{m} \sigma (\nabla_{\tilde{x}^{(c)}} \tau_i) \big(\mathrm{c}(\sigma, \omega, \tau_i) b' - \mathrm{s}(\sigma, \omega, \tau_i) a' \big) A_i'
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=0}^{m}\omega(\nabla_{\tilde{x}^{(c)}}\tau_i)\big(\mathrm{s}(\sigma,\omega,\tau_i)b'+\mathrm{c}(\sigma,\omega,\tau_i)a'\big)A_i' \\
&-\sum_{i=0}^{m}\mathrm{c}(\sigma,\omega,\tau_i)(\nabla_{\tilde{x}^{(c)}}b'A_i')-\mathrm{s}(\sigma,\omega,\tau_i)(\nabla_{\tilde{x}^{(c)}}a'A_i'))\,.
\end{aligned}
$$

The product rule also has to be applied when calculating the Jacobians w.r.t. $\alpha^{(c)}$:

$$
\begin{aligned}
B_{52} =&\nabla_{\alpha^{(c)}}\left(\sigma a'-\omega b'-\sum_{i=0}^{m}\big(\mathrm{c}(\sigma,\omega,\tau_i)a'-\mathrm{s}(\sigma,\omega,\tau_i)b'\big)A_i'\right) \\
=&\sum_{i=0}^{m}\sigma(\nabla_{\alpha^{(c)}}\tau_i)\big(\mathrm{c}(\sigma,\omega,\tau_i)a'+\mathrm{s}(\sigma,\omega,\tau_i)b'\big)A_i' \\
&+\sum_{i=0}^{m}\omega(\nabla_{\alpha^{(c)}}\tau_i)\big(\mathrm{s}(\sigma,\omega,\tau_i)a'-\mathrm{c}(\sigma,\omega,\tau_i)b'\big)A_i' \\
&-\sum_{i=0}^{m}\mathrm{c}(\sigma,\omega,\tau_i)(\nabla_{\alpha^{(c)}}a'A_i')+\mathrm{s}(\sigma,\omega,\tau_i)(\nabla_{\alpha^{(c)}}b'A_i') \\
B_{53} =&\nabla_{\alpha^{(c)}}\left(\omega a'+\sigma b'-\sum_{i=0}^{m}\big(\mathrm{c}(\sigma,\omega,\tau_i)b'+\mathrm{s}(\sigma,\omega,\tau_i)a'\big)A_i'\right) \\
=&\sum_{i=0}^{m}\sigma(\nabla_{\alpha^{(c)}}\tau_i)\big(\mathrm{c}(\sigma,\omega,\tau_i)b'-\mathrm{s}(\sigma,\omega,\tau_i)a'\big)A_i' \\
&+\sum_{i=0}^{m}\omega(\nabla_{\alpha^{(c)}}\tau_i)\big(\mathrm{s}(\sigma,\omega,\tau_i)b'+\mathrm{c}(\sigma,\omega,\tau_i)a'\big)A_i' \\
&-\sum_{i=0}^{m}\mathrm{c}(\sigma,\omega,\tau_i)(\nabla_{\alpha^{(c)}}b'A_i')-\mathrm{s}(\sigma,\omega,\tau_i)(\nabla_{\alpha^{(c)}}a'A_i')
\end{aligned}
$$

It remains to state expressions for the vector matrix products such as $\nabla_{\tilde{x}^{(c)}}a'A_i'$. In order to do so we have to switch to components. The matrix A_i contains

$$
(A_i)_{\rho,\nu}=\frac{\partial f_\rho}{\partial\tilde{x}_\nu^{(c)}(t-\tau_i)}
$$

in its ρ-th row and ν-th column. Multiplying it with a vector a from the right results in $A_i a$ with components

$$
(A_i a)_\nu=\sum_{\rho=1}^{n}a_\rho\frac{\partial f_\nu}{\partial\tilde{x}_\rho^{(c)}(t-\tau_i)}\,.
$$

Transposing and calculating the required derivative yields the matrix $\nabla_{\tilde{x}^{(c)}} a' A_i'$ with components

$$(\nabla_{\tilde{x}^{(c)}} a' A_i')_{\mu,\nu} = \sum_{\rho=1}^{n} a_\rho \frac{\partial^2 f_\nu}{\partial \tilde{x}_\mu^{(c)} \, \partial \tilde{x}_\rho^{(c)}(t-\tau_i)}$$

in its μ-th row and ν-th column. The other derivatives of vector matrix products can be found accordingly.

The matrix whose columns span the normal vector space is now completely determined. We are looking for the parameter space normal vector space, thus we have to find the kernel κ of the first $3n+1$ rows of the transposed Jacobian,

$$\begin{bmatrix} \nabla_{\tilde{x}^{(c)}} f' & B_{12} & B_{13} & 0 & 0 \\ 0 & B_{22} & B_{23} & 2a & b \\ 0 & B_{32} & B_{33} & 2b & a \\ 0 & B_{42} & B_{43} & 0 & 0 \end{bmatrix} \kappa = 0$$

which leads to (6b). By multiplying κ with the last n_α rows of B we get the parameter space component of the normal space,

$$r = \begin{bmatrix} \nabla_{\alpha^{(c)}} f' & B_{52} & B_{53} & 0 & 0 \end{bmatrix} \kappa \,,$$

which yields (6c). The length of r in not determined yet, (6d) fixes the length of r to unit length. □

References

[1] J. Otten and M. Mönnigmann, "Robust optimization of delay differential equations with state and parameter dependent delays," in *Proceedings of the 55th IEEE Conference on Decision and Control*, 2016.

[2] K. Engelborghs, T. Luzyanina, and D. Roose, "Numerical bifurcation analysis of delay differential equations," *Computational and Applied Mathematics*, vol. 125, pp. 265–275, 2000.

[3] M. Mönnigmann and W. Marquardt, "Normal vectors on manifolds of critical points for parametric robustness of equilibrium solutions of ODE systems," *Journal of Nonlinear Science*, vol. 12, pp. 85–112, 2002.